arXiv:1112.6176v1 [math.CA] 28 Dec 2011

THE HERMITE-HADAMARD'S INEQUALITY FOR SOME CONVEX FUNCTIONS VIA FRACTIONAL INTEGRALS AND RELATED RESULTS

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Abstract. In this paper, firstly we have established Hermite-Hadamard's inequalities for $s-$convex functions in the second sense and $m-$convex functions via fractional integrals. Secondly, a Hadamard type integral inequality for the fractional integrals are obtained and these result have some relationships with [11, Theorem 1, page 28-29].

1. Introduction

Let real function f be defined on some nonempty interval I of real line $\mathbb{R}$. The function f is said to be convex on I if inequality

$$f(\lambda x+(1-\lambda)y)\leq \lambda f(x)+(1-\lambda)f(y)$$

holds for all $x,y\in I$ and $\lambda\in[0,1]$.

In [10], Hudzik and Maligranda considered, among others, the class of functions which are $s-$convex in the second sense. This class of functions is defined as the following:

Definition 1. *A function $f:[0,\infty)\rightarrow\mathbb{R}$ is said to be $s-$convex in the second sense if*

$$f(\lambda x+(1-\lambda)y)\leq \lambda^{s}f(x)+(1-\lambda)^{s}f(y)$$

for all $x,y\in[0,\infty)$, $\lambda\in[0,1]$ and for some fixed $s\in(0,1]$. This class of s-convex functions is usually denoted by K_s^2.

It can be easily seen that for $s=1$, $s-$convexity reduces to ordinary convexity of functions defined on $[0,\infty)$.

In [12], G. Toader considered the class of $m-$convex functions: another intermediate between the usual convexity and starshaped convexity.

Definition 2. *The function $f:[0,b]\rightarrow\mathbb{R}$, $b>0$, is said to be $m-$convex, where $m\in[0,1]$, if we have*

$$f(tx+m(1-t)y)\leq tf(x)+m(1-t)f(y)$$

for all $x,y\in[0,b]$ and $t\in[0,1]$. We say that f is $m-$concave if $(-f)$ is $m-$convex.

2000 *Mathematics Subject Classification.* 26D07, 26D10, 26D15, 26A33.

Key words and phrases. $s-$convex functions, Hadamard's inequality, Riemann-Liouville fractional integral.

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Obviously, for $m = 1$ Definition 2 recaptures the concept of standard convex functions on $[a, b]$, and for $m = 0$ the concept starshaped functions.

One of the most famous inequalities for convex functions is Hadamard's inequality. This double inequality is stated as follows (see for example [14] and [5]): Let f be a convex function on some nonempty interval $[a, b]$ of real line $\mathbb{R}$, where $a \neq b$. Then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

Both inequalities hold in the reversed direction if f is concave. We note that Hadamard's inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality. Hadamard's inequality for convex functions has received renewed attention in recent years and a remarkable variety of refinements and generalizations have been found (see, for example, [1]-[16]).

In [8], Hadamard's inequality which for $s-$convex functions in the second sense is proved by S.S. Dragomir and S. Fitzpatrick.

Theorem 1. *Suppose that $f : [0,\infty) \to [0,\infty)$ is an s-convex function in the second sense, where $s \in (0, 1)$, and let $a, b \in [0,\infty)$, $a < b$. If $f \in L^1([a, b])$, then the following inequalities hold:*

$$2^{s-1} f(\frac{a+b}{2}) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{s+1}. \tag{1.2}$$

The constant $k = \frac{1}{s+1}$ is the best possible in the second inequality in (1.2).

In [11], Kirmaci et. al. established a new Hadamard-type inequality which holds for s-convex functions in the second sense. It is given in the next theorem.

Theorem 2. *Let $f : I \to \mathbb{R}$, $I \subset [0,\infty)$, be a differentiable function on I° such that $f' \in L_1([a, b])$, where $a, b \in I$, $a < b$. If $|f'|^q$ is $s-$convex on $[a, b]$ for some fixed $s \in (0, 1)$ and $q \geq 1$, then:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{2}\left(\frac{1}{2}\right)^{\frac{q-1}{q}}\left[\frac{s+\left(\frac{1}{2}\right)^s}{(s+1)(s+2)}\right]^{\frac{1}{q}}\left[|f'(a)|^q + |f'(b)|^q\right]^{\frac{1}{q}}. \tag{1.3}$$

We give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used throughout this paper.

Definition 3. *Let $f \in L_1[a, b]$. The Riemann-Liouville integrals $J^\alpha_{a+}f$ and $J^\alpha_{b-}f$ of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$J^\alpha_{a+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a$$

and

$$J^\alpha_{b-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b$$

respectively where $\Gamma(\alpha) = \int_0^\infty e^{-t}u^{\alpha-1}du$. Here is $J^0_{a+}f(x) = J^0_{b-}f(x) = f(x)$.

In the case of $\alpha = 1$, the fractional integral reduces to the classical integral. Properties concerning this operator can be found ([23],[24] and [25]).

For some recent results connected with fractional integral inequalities see ([17]-[27])

In [27] Sarıkaya et. al. proved a variant of the identity is established by Dragomir and Agarwal in [6, Lemma 2.1] for fractional integrals as the following:

Lemma 1. *Let $f:[a,b]\to\mathbb{R}$, be a differentiable mapping on (a,b) with $a<b$. If $f'\in L[a,b]$, then the following equality for fractional integrals holds:*

$$\begin{aligned}&\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]\\ &= \frac{b-a}{2}\int_0^1\left[(1-t)^{\alpha}-t^{\alpha}\right]f'(ta+(1-t)b)\,dt.\end{aligned} \tag{1.4}$$

The aim of this paper is to establish Hadamard's inequality and Hadamard type inequalities for $s-$convex functions in the second sense and $m-$conex functions via Riemann-Liouville fractional integral.

2. Hermite-Hadamard Type Inequalities for some convex functions via Fractional Integrals

2.1. **For $s-$convex functions.** Hadamard's inequality can be represented for $s-$convex functions in fractional integral forms as follows:

Theorem 3. *Let $f:[a,b]\to\mathbb{R}$ be a positive function with $0\le a<b$ and $f\in L_1[a,b]$. If f is a $s-$convex mapping in the second sense on $[a,b]$, then the following inequalities for fractional integrals with $\alpha>0$ and $s\in(0,1)$ hold:*

$$2^{s-1}f\left(\frac{a+b}{2}\right)\le\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[\frac{J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)}{2}\right]\le\left[\frac{1}{(\alpha+s)}+\beta(\alpha,s+1)\right]\frac{f(a)+f(b)}{2} \tag{2.1}$$

where β is Euler Beta function.

Proof. Since f is a $s-$convex mapping in the second sense on $[a,b]$, we have for $x,y\in[a,b]$ with $\lambda=\frac{1}{2}$

$$f\left(\frac{x+y}{2}\right)\le\frac{f(x)+f(y)}{2^s}. \tag{2.2}$$

Now, let $x=ta+(1-t)b$ and $y=(1-t)a+tb$ with $t\in[0,1]$. Then, we get by (2.2) that:

$$2^s f\left(\frac{a+b}{2}\right)\le f(ta+(1-t)b)+f((1-t)a+tb) \tag{2.3}$$

for all $t\in[0,1]$.

Multiplying both sides of (2.3) by $t^{\alpha-1}$, then integrating the resulting inequality with respest to t over $[0,1]$, we obtain

$$\begin{aligned}
&\frac{2^s}{\alpha} f\left(\frac{a+b}{2}\right) \\
\leq &\int_0^1 t^{\alpha-1} f\left(ta+(1-t)b\right) dt + \int_0^1 t^{\alpha-1} f\left((1-t)a+tb\right) dt \\
= &\frac{1}{(b-a)^\alpha} \int_a^b (b-u)^{\alpha-1} f(u)du - \frac{1}{(a-b)^\alpha} \int_a^b (a-v)^{\alpha-1} f(v)dv \\
= &\frac{\Gamma(\alpha)}{(b-a)^\alpha} \left[J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)\right]
\end{aligned}$$

i.e.

$$2^{s-1} f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{(b-a)^\alpha} \left[\frac{J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)}{2}\right]$$

and the first inequality is proved.

For the proof of the second inequality in (2.1) we first note that if f is a $s-$convex mapping in the second sense, then, for $t \in [0,1]$, it yields

$$f\left(ta+(1-t)b\right) \leq t^s f(a) + (1-t)^s f(b)$$

and

$$f\left((1-t)a+tb\right) \leq (1-t)^s f(a) + t^s f(b).$$

By adding these inequalities we have

$$f\left(ta+(1-t)b\right) + f\left((1-t)a+tb\right) \leq \left[t^s + (1-t)^s\right]\left(f(a)+f(b)\right). \tag{2.4}$$

Thus, multiplying both sides of (2.4) by $t^{\alpha-1}$and integrating the resulting inequality with respest to t over $[0,1]$, we obtain

$$\begin{aligned}
&\int_0^1 t^{\alpha-1} f\left(ta+(1-t)b\right) dt + \int_0^1 t^{\alpha-1} f\left((1-t)a+tb\right) dt \\
\leq &\left[f(a)+f(b)\right] \int_0^1 t^{\alpha-1}\left[t^s + (1-t)^s\right] dt
\end{aligned}$$

i.e.

$$\frac{\Gamma(\alpha)}{(b-a)^\alpha} \left[J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)\right] \leq \left[1 + (\alpha+s)\,\beta(\alpha, s+1)\right] \frac{f(a)+f(b)}{(\alpha+s)}$$

where the proof is completed. □

Remark 1. *If we choose $\alpha = 1$ in Theorem 3, then the inequalities (2.1) become the inequalities (1.2) of Theorem 1.*

Using Lemma 1, we can obtain the following fractional integral inequality for $s-$convex in the second sense:

Theorem 4. *Let $f : [a,b] \subset [0,\infty) \to \mathbb{R}$ be a differentiable mapping on (a,b) with $a < b$ such that $f' \in L[a,b]$. If $|f'|^q$ is $s-$convex in the second sense on $[a,b]$ for*

some fixed $s \in (0,1)$ and $q \geq 1$, then the following inequality for fractional integrals holds:

$$\begin{aligned}
&\left| \frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a) \right] \right| \\
&\leq \frac{b-a}{2} \left[\frac{2}{\alpha+1} \left(1 - \frac{1}{2^{\alpha}} \right) \right]^{\frac{q-1}{q}} \\
&\times \left\{ \beta\left(\frac{1}{2}; s+1, \alpha+1\right) - \beta\left(\frac{1}{2}; \alpha+1, s+1\right) + \frac{2^{\alpha+s}-1}{(\alpha+s+1)\,2^{\alpha+s}} \right\} \left(|f'(a)|^{q} + |f'(b)|^{q} \right)^{\frac{1}{q}}.
\end{aligned} \tag{2.5}$$

Proof. Suppose that $q = 1$. From Lemma 1 and using the properties of modulus, we have

$$\begin{aligned}
&\left| \frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a) \right] \right| \\
&\leq \frac{b-a}{2} \int_0^1 |(1-t)^{\alpha} - t^{\alpha}| \, |f'(ta+(1-t)b)| \, dt.
\end{aligned} \tag{2.6}$$

Since $|f'|$ is $s-$convex on $[a,b]$, we have

$$\begin{aligned}
&\left| \frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a) \right] \right| \\
&\leq \frac{b-a}{2} \int_0^1 |(1-t)^{\alpha} - t^{\alpha}| \left[t^{s} |f'(a)| + (1-t)^{s} |f'(b)| \right] dt \\
&= \frac{b-a}{2} \left\{ \int_0^{\frac{1}{2}} \left[(1-t)^{\alpha} - t^{\alpha} \right] \left[t^{s} |f'(a)| + (1-t)^{s} |f'(b)| \right] dt \right. \\
&\quad \left. + \int_{\frac{1}{2}}^1 \left[t^{\alpha} - (1-t)^{\alpha} \right] \left[t^{s} |f'(a)| + (1-t)^{s} |f'(b)| \right] dt \right\} \\
&= \frac{b-a}{2} \left\{ |f'(a))| \int_0^{\frac{1}{2}} t^{s} (1-t)^{\alpha} dt - |f'(a)| \int_0^{\frac{1}{2}} t^{s+\alpha} dt \right. \\
&\quad + |f'(b))| \int_0^{\frac{1}{2}} (1-t)^{s+\alpha} dt - |f'(b))| \int_0^{\frac{1}{2}} (1-t)^{s} t^{\alpha} dt \\
&\quad + |f'(a))| \int_{\frac{1}{2}}^1 t^{\alpha+s} dt - |f'(a))| \int_{\frac{1}{2}}^1 t^{s} (1-t)^{\alpha} dt \\
&\quad + |f'(b))| \int_{\frac{1}{2}}^1 (1-t)^{s} t^{\alpha} dt - |f'(b))| \int_{\frac{1}{2}}^1 (1-t)^{s+\alpha} dt.
\end{aligned} \tag{2.7}$$

Since

$$\begin{aligned}
\int_0^{\frac{1}{2}} t^s (1-t)^\alpha \, dt &= \int_{\frac{1}{2}}^1 (1-t)^s t^\alpha dt = \beta\left(\frac{1}{2}; s+1, \alpha+1\right), \\
\int_0^{\frac{1}{2}} (1-t)^s t^\alpha dt &= \int_{\frac{1}{2}}^1 t^s (1-t)^\alpha \, dt = \beta\left(\frac{1}{2}; \alpha+1, s+1\right), \\
\int_0^{\frac{1}{2}} t^{s+\alpha} dt &= \int_{\frac{1}{2}}^1 (1-t)^{s+\alpha} \, dt = \frac{1}{2^{s+\alpha+1}(s+\alpha+1)}
\end{aligned}$$

and

$$\int_0^{\frac{1}{2}} (1-t)^{s+\alpha} \, dt = \int_{\frac{1}{2}}^1 t^{s+\alpha} dt = \frac{1}{(s+\alpha+1)} - \frac{1}{2^{s+\alpha+1}(s+\alpha+1)}.$$

We obtain

$$\begin{aligned}
&\left| \frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left[J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a) \right] \right| \\
\leq\ & \frac{b-a}{2} \left[|f'(a)| + |f'(b)| \right] \left\{ \beta\left(\frac{1}{2}; s+1, \alpha+1\right) - \beta\left(\frac{1}{2}; \alpha+1, s+1\right) + \frac{2^{\alpha+s}-1}{(\alpha+s+1)\,2^{\alpha+s}} \right\}
\end{aligned}$$

which completes the proof for this case. Suppose now that $q > 1$. Since $|f'|^q$ is $s-$convex on $[a,b]$, we know that for every $t \in [0,1]$

$$|f'(ta+(1-t)b)|^q \leq t^s |f'(a)|^q + (1-t)^s |f'(b)|^q, \tag{2.8}$$

so using well know Hölder's inequality (see for example [?]) for $\frac{1}{p} + \frac{1}{q} = 1$, ($q > 1$) and (2.8) in (2.6), we have successively

$$\begin{aligned}
&\left| \frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left[J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a) \right] \right| \\
\leq\ & \frac{b-a}{2} \int_0^1 |(1-t)^\alpha - t^\alpha| \, |f'(ta+(1-t)b)| \, dt \\
=\ & \frac{b-a}{2} \int_0^1 |(1-t)^\alpha - t^\alpha|^{1-\frac{1}{q}} \, |(1-t)^\alpha - t^\alpha|^{\frac{1}{q}} \, |f'(ta+(1-t)b)| \, dt \\
\leq\ & \frac{b-a}{2} \left(\int_0^1 |(1-t)^\alpha - t^\alpha| \, dt \right)^{\frac{q-1}{q}} \left(\int_0^1 |(1-t)^\alpha - t^\alpha| \, |f'(ta+(1-t)b)|^q \, dt \right)^{\frac{1}{q}} \\
\leq\ & \frac{b-a}{2} \left[\frac{2}{\alpha+1} \left(1 - \frac{1}{2^\alpha} \right) \right]^{\frac{q-1}{q}} \\
& \times \left\{ \beta\left(\frac{1}{2}; s+1, \alpha+1\right) - \beta\left(\frac{1}{2}; s+1, \alpha+1\right) + \frac{2^{\alpha+s}-1}{(\alpha+s+1)\,2^{\alpha+s}} \right\} \left(|f'(a)|^q + |f'(b)|^q \right)^{\frac{1}{q}}
\end{aligned}$$

where we use the fact that

$$\int_0^1 |(1-t)^\alpha - t^\alpha| \, dt = \int_0^{\frac{1}{2}} \left[(1-t)^\alpha - t^\alpha \right] dt + \int_{\frac{1}{2}}^1 \left[t^\alpha - (1-t)^\alpha \right] dt = \frac{2}{\alpha+1} \left(1 - \frac{1}{2^\alpha} \right)$$

which completes the proof. □

Remark 2. *If we take $\alpha = 1$ in Theorem 4, then the inequality (2.5) becomes the inequality (1.3) of Theorem 2.*

2.2. **For $m-$convex functions.** We start with the following theorem:

Theorem 5. *Let $f : [0, \infty] \to \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L_1[a, b]$. If f is $m-convex$ mapping on $[a, b]$, then the following inequalities for fractional integral with $\alpha > 0$ and $m \in (0, 1]$ hold:*

$$\begin{aligned}\frac{2}{\Gamma(\alpha+1)} f\left(\frac{m\,(a+b)}{2}\right) &\leq \frac{1}{(mb-ma)^{\alpha}} J^{\alpha}_{(ma)^{+}} f(mb) + \frac{m}{(b-a)^{\alpha}} J^{\alpha}_{b^{-}} f(a) \\ &\leq \frac{f(ma) + m^2 f(\frac{b}{m})}{(\alpha+1)} + m\frac{f(a)+f(b)}{\alpha(\alpha+1)}\end{aligned} \tag{2.9}$$

Proof. Since f is $m-convex$ functions, we have

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y)$$

and if we choose $t = \frac{1}{2}$ we get

$$f\left(\frac{1}{2}(x+my)\right) \leq \frac{f(x)+mf(y)}{2}.$$

Now, let $x = mta + m(1-t)b$ and $y = (1-t)a + tb$ with $t \in [0, 1]$. Then we get

$$\begin{aligned} f\left(\frac{1}{2}(mta + m(1-t)b + m(1-t)a + mtb)\right) &\leq \frac{f(mta + m(1-t)b) + mf((1-t)a+tb)}{2} \\ f\left(\frac{1}{2}m\,(a+b)\right) &\leq \frac{f(mta + m(1-t)b) + mf((1-t)a+tb)}{2}. \qquad (2.10)\end{aligned}$$

Multiplying both sides of (2.10) by $t^{\alpha-1}$, then integrating the resulting inequlity with respect to t over $[0, 1]$, we obtain

$$\begin{aligned} f\left(\frac{1}{2}m\,(a+b)\right)\int_0^1 t^{\alpha-1}dt &\leq \frac{1}{2}\int_0^1 t^{\alpha-1} f(mta + m(1-t)b)dt + \frac{m}{2}\int_0^1 t^{\alpha-1} f((1-t)a+tb)dt \\ \frac{1}{\alpha} f\left(\frac{1}{2}m\,(a+b)\right) &\leq \frac{1}{2}\int_{mb}^{ma}\left(\frac{u-mb}{ma-mb}\right)^{\alpha-1} f(u)\frac{du}{m(a-b)} + \frac{m}{2}\int_a^b \left(\frac{v-a}{b-a}\right)^{\alpha-1} f(v)\frac{dv}{b-a} \\ &\leq \frac{1}{2(mb-ma)^{\alpha}}\int_{ma}^{mb}(mb-u)^{\alpha-1} f(u)du + \frac{m}{2}\frac{\Gamma(\alpha)}{(b-a)^{\alpha}} J^{\alpha}_{b^{-}} f(a)\end{aligned}$$

which the first inequality is proved.

By the m-convexity of f, we also have

$$\begin{aligned} &\frac{1}{2}\left[f(mta + m(1-t)b) + mf((1-t)a+tb)\right] \\ \leq\ &\frac{1}{2}\left[mtf(a) + m(1-t)f(b) + m(1-t)f(a) + m^2 f(\frac{b}{m})\right]\end{aligned}$$

for all $t \in [0,1]$. Multiplying both sides of above inequality by $t^{\alpha-1}$ and integrating over $t \in [0,1]$, we get

$$\begin{aligned}
&\frac{1}{(mb-ma)^{\alpha}}\int_{ma}^{mb}(mb-u)^{\alpha-1}f(u)du+\frac{m}{(b-a)^{\alpha}}\int_{a}^{b}(v-a)^{\alpha-1}f(v)dv \\
\leq\ & \frac{f(ma)+m^{2}f(\frac{b}{m})}{(\alpha+1)}+m\frac{f(a)+f(b)}{\alpha(\alpha+1)}
\end{aligned}$$

which this gives the second part of (2.9). □

Corollary 1. *Under the conditions in Theorem 5 with $\alpha = 1$, then the following inequality hold:*

$$\begin{aligned}
f\left(\frac{m\,(a+b)}{2}\right) &\leq \frac{1}{(b-a)}\int_{a}^{b}\frac{f(mx)+mf(x)}{2}dx \\
&\leq \frac{1}{2}\left[\frac{f(ma)+m^{2}f(\frac{b}{m})}{2}+m\frac{f(a)+f(b)}{2}\right].
\end{aligned} \tag{2.11}$$

Remark 3. *If we take $m = 1$ in Corollary 1, then the inequalities (2.11) become the inequalities (1.1).*

Theorem 6. *Let $f : [0,\infty] \to \mathbb{R}$, be m-convex functions with $m \in (0,1]$, $0 \leq a < b$ and $f \in L_1[a,b]$. $F(x,y)_{(t)} : [0,1] \to \mathbb{R}$ are defined as the following:*

$$F(x,y)_{(t)}=\frac{1}{2}[f(tx+m(1-t)y)+f((1-t)x+mty)].$$

Then, we have

$$\frac{1}{(b-a)^{\alpha}}\int_{a}^{b}(b-u)^{\alpha-1}F\left(u,\frac{a+b}{2}\right)_{\left(\frac{b-u}{b-a}\right)}du\leq\frac{\Gamma(\alpha)}{2(b-a)^{\alpha}}J_{a^{+}}^{\alpha}f(b)+\frac{m}{2\alpha}f\left(\frac{a+b}{2}\right)$$

for all $t \in [0,1]$.

Proof. Since f and g are $m-convex$ functions, we have

$$\begin{aligned}
F(x,y)_{(t)} &\leq \frac{1}{2}\left[tf(x)+m(1-t)f(y)+(1-t)f(x)+mtf(y)\right] \\
&= \frac{1}{2}\left[f(x)+mf(y)\right]
\end{aligned}$$

and so,

$$F\left(x,\frac{a+b}{2}\right)_{(t)}\leq\frac{1}{2}\left[f(x)+mf\left(\frac{a+b}{2}\right)\right].$$

If we choose $x = ta + (1-t)b$, we have

$$F\left(ta+(1-t)b,\frac{a+b}{2}\right)_{(t)}\leq\frac{1}{2}\left[f(ta+(1-t)b)+mf\left(\frac{a+b}{2}\right)\right]. \tag{2.12}$$

Thus multiplying both sides of (2.12) by $t^{\alpha-1}$, then integrating the resulting inequality with respect to t over $[0,1]$, we obtain

$$\int_0^1 t^{\alpha-1} F\left(ta+(1-t)b, \frac{a+b}{2}\right)_{(t)} dt \leq \frac{1}{2}\left[\int_0^1 t^{\alpha-1} f(ta+(1-t)b)dt + \int_0^1 t^{\alpha-1} mf\left(\frac{a+b}{2}\right) dt\right].$$

Thus, if we use the change of the variable $u = ta+(1-t)b,\ t \in [0,1]$, then have the conclusion. □

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